

## B Sc Part I Sem II Paper I Unit IV

### Exponential Distribution

**Definition** – A continuous r v is said to follow Exponential Distribution

With parameter  $\theta$  if its p d f is given by

$$f(x) = \theta e^{-\theta x}, \quad x > 0$$
$$= 0 \quad \text{otherwise}$$

To check whether  $f(x)$  is the p d f

$$\int_0^{\infty} f(x) dx = \int_0^{\infty} \theta e^{-x\theta} dx = 1$$

To find Cumulative distribution function ( C D F )

$$F(x) = P[X \leq x] = \int_0^x \theta e^{-x\theta} dx = \theta \left[ \frac{e^{-\theta x}}{-\theta} \right] \quad \{ \text{limits from 0 to } x \}$$
$$F(x) = 1 - e^{-\theta x}$$

To find the moment generating function

$$M_x(t) = E[e^{tX}] = \int_0^{\infty} \theta e^{tX} e^{-x\theta} dx$$
$$= \theta \int_0^{\infty} e^{tX} e^{-x\theta} dx = \theta \int_0^{\infty} e^{-(\theta-t)x} dx$$
$$= [\theta / (\theta-t)] = 1 / (1 - t/\theta) = (1 - t/\theta)^{-1}$$
$$= 1 + t/\theta + t^2/\theta^2 + t^3/\theta^3 + t^4/\theta^4 + \dots$$

$\mu_r^1 =$  Coefficient of  $t^r / r!$  in the expression of  $M_x(t)$

The  $r^{\text{th}}$  term =  $[t^r / r!][r! / \theta^r]$  multiply and divide by  $r!$

$$\mu_r' = r! / \theta^r$$

Giving different values to r

$$r = 1, \mu'_1 = \text{mean} = 1 / \theta$$

$$r = 2, \mu'_2 = 2! / \theta^2 = 2 / \theta^2$$

$$\text{Variance} = \mu'_2 - [\mu'_1]^2 = 1 / \theta^2$$

$$\text{Standard Deviation} = 1 / \theta$$

Therefore for Exponential Distribution

$$\text{Mean} = \text{Standard Deviation} = 1 / \theta$$

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### Lack of memory property of Exponential Distribution

Example : The life time of an electric bulb follows Exponential Distribution .It can be seen that the conditional probability that the electric bulb runs for at least ( a+b) units of time given that it has already run for “a” units of time is same as the probability that it runs for at least “b” units of time. That it does not take into consideration the information that it has already run for “a” units of time. In this sense it is said that Exponential Distribution lacks in memory.

To show that  $P[X > a+b / X > a] = P[X > b]$

$$\begin{aligned} P[X > a+b / X > a] &= P[X > a+b \cap X > a] / P[X > a] \\ &= P[X > a+b] / P[X > a] \quad \text{-----(1)} \end{aligned}$$

Now distribution function for expo dist is  $F(x) = 1 - e^{-\theta x}$

Now consider the Numerator in equation (1)

$$P[X > a+b] = 1 - P[X \leq a+b] = 1 - F(a+b)$$

$$= 1 - [1 - e^{-\theta(a+b)}]$$

$$= e^{-\theta(a+b)}$$

Similarly Denominator  $P[X > a] = e^{-a\theta}$

$$\text{Therefore } P[X > a+b \mid X > a] = e^{-\theta(a+b)} / e^{-a\theta}$$

$$= e^{-b\theta}$$

$$= P[X > b] = e^{-\theta(a+b)}$$

## Gamma Distribution

There are two types of Gamma Distributions depending upon the number of parameters.

- 1) One parameter Gamma Distribution
- 2) Two parameter Gamma Distribution

### One parameter Gamma Distribution

**Definition :** A continuous R V X is said to follow Gamma Distribution

With parameter m if its p.d.f. is given by

$$f(x) = \frac{1}{\Gamma(m)} e^{-x} x^{m-1} \quad ; x > 0$$

$$= 0 \quad \text{otherwise}$$

We denote this as  $x \sim \gamma(m)$

The integral  $\int_0^{\infty} e^{-x} x^{m-1} dx$  is called gamma integral whose value is  $\Gamma(m)$ .

$$\text{Therefore } \int_0^{\infty} e^{-x} x^{m-1} dx = \Gamma(m)$$

$$\frac{1}{\Gamma m} \int_0^{\infty} e^{-x} x^{m-1} dx = 1 \Rightarrow \int_0^{\infty} f(x) dx = 1$$

$\Rightarrow f(x)$  is a p.d.f.

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To find m g f of one parameter Gamma Distribution

$$\begin{aligned} M_x(t) &= E [e^{tx}] = \int_0^{\infty} e^{tx} f(x) dx \\ &= \frac{1}{\Gamma m} \int_0^{\infty} e^{tx} e^{-x} x^{m-1} dx \\ &= \frac{1}{\Gamma m} \int_0^{\infty} e^{-(1-t)x} x^{m-1} dx \\ &= \frac{1}{\Gamma m} \frac{\Gamma^m}{(1-t)^{-m}} \end{aligned}$$

$$M_x(t) = (1-t)^{-m}$$

To find moments using m g f

$$\mu'_1 = \left[ \frac{d}{dt} M_x(t) \right] \text{ at } t=0$$

$$\text{Mean} = \mu'_1 = m$$

$$\mu'_2 = \left[ \frac{d^2}{dt^2} M_x(t) \right] \text{ at } t=0 \text{ gives } m(m+1)$$

$$\mu'_2 = m(m+1)$$

$$\text{Variance} = \mu'_2 - [\mu'_1]^2$$

$$= m(m+1) - m^2 = m$$

For one parameter Gamma Distribution Mean=Variance=m

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## Additivity Property of the one parameter Gamma Distribution

If  $X_1, X_2, \dots, X_n$  are  $n$  independent gamma variates with parameters  $m_1, m_2, \dots, m_n$  respectively then  $\sum X_i$  follows gamma distribution with parameter  $\sum m_i$  i.e.  $\sum X_i \sim \gamma(\sum m_i)$

Proof - Given  $X_i \sim \gamma(m_i)$

$$\text{Therefore } M_{X_i}(t) = (1-t)^{-m_i}, i=1, 2, 3, \dots, n$$

$$\begin{aligned} M_{\sum X_i}(t) &= M_{X_1}(t) M_{X_2}(t) \dots M_{X_n}(t) \\ &= (1-t)^{-m_1} (1-t)^{-m_2} \dots (1-t)^{-m_n} \\ &= (1-t)^{-\sum m_i} \end{aligned}$$

Which is the m.g.f of  $\gamma$  variate with parameter  $\sum m_i$  i.e.

$$\sum X_i \sim \gamma(\sum m_i)$$

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## Two parameter Gamma Distribution

**Definition :** A continuous R.V.  $X$  is said to follow Gamma Distribution

With parameters  $a$  and  $m$  if its p.d.f. is given by

$$\begin{aligned} f(x) &= \frac{a^m}{\Gamma(m)} e^{-ax} x^{m-1}; x > 0 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

We denote this as  $x \sim \gamma(a, m)$

If we put  $a=1$  in this p d f ,we get the pdf of one parameter gamma distribution. So one parameter gamma distribution is a particular case of Two parameter Gamma Distribution

Also the integral  $\int_0^{\infty} e^{-ax} x^{m-1} dx$  is the gamma integral with value

$$\frac{a^m}{\Gamma m}$$

Therefore  $\frac{a^m}{\Gamma m} \int_0^{\infty} e^{-ax} x^{m-1} dx = 1$

$$\Rightarrow \int_0^{\infty} f(x)dx = 1$$

$\Rightarrow f(x)$  is a p.d.f.

To find m g f

$$\begin{aligned} M_x(t) &= E [e^{tX}] = \int_0^{\infty} e^{tx} f(x) dx \\ &= \int_0^{\infty} e^{tx} \frac{a^m}{\Gamma m} e^{-ax} x^{m-1} dx \\ &= \frac{a^m}{\Gamma m} \int_0^{\infty} e^{-(a-t)x} x^{m-1} dx \\ &= \frac{a^m}{\Gamma m} \frac{\Gamma m}{(a-t)^m} \\ M_x(t) &= \left(\frac{a}{a-t}\right)^m \\ &= \left(\frac{1}{1-\frac{t}{a}}\right)^m = (1 - t/a)^{-m} \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} M_x(t) &= -m (1 - t/a)^{-m-1} \left(-\frac{1}{a}\right) \\ &= \frac{m}{a} (1 - t/a)^{-m-1} \end{aligned}$$

$$\mu'_1 = \left[ \frac{d}{dt} M_x(t) \right] \text{ at } t = 0 = \frac{m}{a}$$

$$\begin{aligned} \mu'_2 &= \left[ \frac{d^2}{dt^2} M_x(t) \right] \text{ at } t = 0 \\ &= \frac{d}{dt} \left[ \frac{m}{a} (1 - t/a)^{-m-1} \right] = \frac{m}{a} \left( \frac{m+1}{a} \right) (1 - t/a)^{-m-2} \end{aligned}$$

$$\mu'_2 = \left[ \frac{d^2}{dt^2} M_x(t) \right] \text{ at } t = 0 = \frac{m(m+1)}{a^2}$$

$$\text{Variance} = \mu'_2 - [\mu'_1]^2 = \frac{m(m+1)}{a^2} - \left( \frac{m}{a} \right)^2$$

$$\text{Variance} = \mu'_2 - [\mu'_1]^2 = \frac{m}{a}$$

Examples - 1) If the m g f of a r v X is  $(1 - t)^{-4}$

Identify the distribution and state its p d f , mean and variance

2) If the m g f of a r v X is

i)  $(1 - t/2)^{-3}$  Identify the distribution and state its p d f , mean and variance.

ii)  $(1 - 3t)^{-5}$  Identify the distribution and state its p d f , mean and variance.

### Additivity Property of the two parameter Gamma Distribution

If  $X_1, X_2, \dots, X_n$  are n independent gamma variates with parameters

$(a, m_1), (a, m_2), \dots, (a, m_n)$  respectively then  $\sum X_i$  follows gamma distribution with parameter  $(a, \sum m_i)$

ie  $\sum X_i \sim \gamma(a, \sum m_i)$

Proof - Given  $X_i \sim \gamma(a, m_i)$

Therefore  $M_{X_i}(t) = (1 - t/a)^{-m_i}, i= 1, 2, 3, \dots, n$

$$\begin{aligned} M_{\sum X_i}(t) &= M_{X_1}(t) M_{X_2}(t) \dots M_{X_n}(t) \\ &= (1 - t/a)^{-m_1} (1 - t/a)^{-m_2} \dots (1 - t/a)^{-m_n} \end{aligned}$$

$$= (1 - t/a)^{-\sum m_i}$$

Which is the m g f of *gamma* variate with parameter  $a$  and  $\sum m_i$

Therefore by uniqueness theorem of m g f

$$\sum X_i \sim \gamma(a, \sum m_i)$$

But if the parameter  $a$  is different for different random variables then  $\sum X_i$  will not follow *gamma* distribution and so the additivity property will not hold good in that case.

## Beta distribution

Beta function – The beta function is defined as  $\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$

The integral  $\int_0^1 x^{m-1} (1-x)^{n-1} dx$

Is called as beta integral of first kind and its value is  $\beta(m, n)$

$$\text{Therefore } \frac{1}{\beta(m, n)} \int_0^1 x^{m-1} (1-x)^{n-1} dx = 1 \text{-----} \quad (1)$$

## Beta distribution of first kind

A continuous R V  $X$  is said to follow Beta Distribution of first kind

If its p.d.f. is given by

$$f(x) = \begin{cases} \frac{1}{\beta(m, n)} x^{m-1} (1-x)^{n-1} & \text{if } 0 < x < 1 \\ = 0 & \text{otherwise} \end{cases}$$

We denote this as  $x \sim \beta_1(m, n)$

## Moments

### The $r^{\text{th}}$ raw moment

$$\begin{aligned}\mu'_r = E[X^r] &= \frac{1}{\beta(m,n)} \int_0^1 x^{m-1} (1-x)^{n-1} x^r dx \\ &= \frac{1}{\beta(m,n)} \int_0^1 x^{(m+r)-1} (1-x)^{n-1} dx \\ &= \frac{1}{\beta(m,n)} \beta(m+r,n) = \\ &= \frac{\Gamma(m+n)}{\Gamma m \Gamma n} \frac{\Gamma(m+r) \Gamma n}{\Gamma(m+n+r)} = \frac{\Gamma(m+n)}{\Gamma m} \frac{\Gamma(m+r)}{\Gamma(m+n+r)} \text{-----}(2)\end{aligned}$$

Put  $r=1 \Rightarrow \mu'_1 = \frac{\Gamma(m+n)}{\Gamma m} \frac{\Gamma(m+1)}{\Gamma(m+n+1)}$ , But  $\Gamma(m+1) = m \Gamma m$

$$\text{Mean} = \mu'_1 = \frac{m}{m+n}$$

$\mu'_2 = E[X^2]$ , putting  $r=2$  in the expression for  $\mu'_r$

$$\begin{aligned}&= \frac{\Gamma(m+n)}{\Gamma m} \frac{\Gamma(m+2)}{\Gamma(m+n+2)} \\ &= \frac{\Gamma(m+n)}{\Gamma m} \frac{m(m+1) \Gamma m}{(m+n)(m+n+1) \Gamma(m+n)} = \frac{m(m+1)}{(m+n)(m+n+1)}\end{aligned}$$

$$\text{Variance} = \mu'_2 - [\mu'_1]^2 = \frac{m(m+1)}{(m+n)(m+n+1)} - \left[ \frac{m}{m+n} \right]^2$$

$$\text{Variance} = \frac{mn}{(m+n)^2(m+n+1)}$$

$$\text{Harmonic Mean} = H = \frac{1}{E\left(\frac{1}{X}\right)}$$

$$\begin{aligned} E\left(\frac{1}{X}\right) &= \frac{1}{\beta(m,n)} \int_0^1 \frac{1}{x} x^{m-1} (1-x)^{n-1} dx \\ &= \frac{1}{\beta(m,n)} \int_0^1 x^{(m-1)-1} (1-x)^{n-1} dx \\ &= \frac{1}{\beta(m,n)} \beta(m-1, n) = \frac{\Gamma(m+n)}{\Gamma m \Gamma n} \frac{\Gamma(m-1) \Gamma n}{\Gamma(m+n+1)} \\ H &= \frac{m-1}{m+n-1} \end{aligned}$$


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## Beta distribution of second kind

A continuous R V X is said to follow Beta Distribution of second kind with parameters m and n if its p.d.f. is given by

$$\begin{aligned} f(x) &= \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{(m+n)}} dx, \quad (m, n) > 0 \quad \text{if } 0 < x < \infty \\ &= 0 \quad \text{otherwise} \end{aligned}$$

We denote this as  $x \sim \beta_2(m, n)$

## Constants of Beta distribution of second kind

$$\begin{aligned} \mu'_r = E[X^r] &= \frac{1}{\beta(m,n)} \int_0^\infty x^r \frac{x^{m-1}}{(1+x)^{(m+n)}} dx \\ &= \frac{1}{\beta(m,n)} \int_0^\infty x^{(r+m)-1} \frac{1}{(1+x)^{(m+r+n-r)}} dx \end{aligned}$$

$$= \frac{1}{\beta(m,n)} \beta(m+r, n-r)$$

$$= \frac{\Gamma^{m+n}}{\Gamma^m \Gamma^n} \frac{\Gamma^{m+r} \Gamma^{n-r}}{\Gamma^{m+n}} = \frac{\Gamma^{m+r}}{\Gamma^m} \frac{\Gamma^{n-r}}{\Gamma^n} \quad n > r$$

Giving different values to r

$$r=1, \quad \mu'_1 = \frac{m}{n-1}$$

$$r=2 \quad \mu'_2 = \frac{m(m+1)}{(n-1)(n-2)}$$

$$\text{Variance} = \mu'_2 - [\mu'_1]^2$$

$$= \frac{m(m+n-1)}{(n-1)^2(n-2)}$$

The harmonic mean H is given by

$$\text{Harmonic Mean} = H = \frac{1}{E\left(\frac{1}{X}\right)}$$

$$\frac{1}{H} = E\left(\frac{1}{X}\right) = \frac{1}{\beta(m,n)} \int_0^\infty \frac{1}{x} \frac{x^{m-1}}{(1+x)^{(m+n)}} dx$$

$$= \frac{1}{\beta(m,n)} \int \frac{x^{m-2}}{(1+x)^{(m+n)}} dx$$

$$= \frac{1}{\beta(m,n)} \int \frac{x^{m-1-1}}{(1+x)^{(m-1+n+1)}} dx$$

$$= \frac{1}{\beta(m,n)} \beta(m-1, n+1) \quad , m > 1$$

On simplification

$$\frac{1}{H} = \frac{m}{n-1} \quad \text{Therefore } H = \frac{n-1}{m}$$

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